

Mathematics 100A Quiz 2

Due: Friday 23 October 2020

Instructions: This is a closed note, closed book quiz. Please write clearly and fully explain your solutions. You have 30 minutes from when you begin to read the questions to complete it.

1. Must every group of order 8 be abelian? If yes, prove that every group of order 8 is abelian. If not, give an example of a non-abelian group of order 8.

Proof. The answer is no. For example, we discussed in lecture the quaternion group of order 8, which is nonabelian. Recall that this group Q is a subgroup of

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} : \alpha, \beta \in \mathbf{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$$

and Q consists of the eight elements

$$Q = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}.$$

□

2. Suppose that G is a group, and H, K are subgroups of G . Define a relation on G as $g \sim g'$ if there exists $h \in H$ and $k \in K$ so that $g' = h g k$. Is this relation an equivalence relation? If so, prove it is an equivalence relation. If not, explain why it is not an equivalence relation.

Proof. Yes, this relation is an equivalence relation. It is transitive, because if $g' = h_1 g k_1$ and $g'' = h_2 g' k_2$ then $g'' = (h_2 h_1) g (k_1 k_2)$. Here $h_1, h_2 \in H$ and $k_1, k_2 \in K$. The relation is symmetric because if $g' = h g k$ then $g = h^{-1} g' k^{-1}$. And the relation is reflexive because $1 \in H$ and $1 \in K$ so $g = (1) g (1)$ so g is related to itself. Note that to check that this is an equivalence relations we are using that because H and K are subgroups: for the transitivity we use that H and K are closed under multiplication, for the symmetry we use that H and K are closed under inverses, and for the reflexivity we use that H and K contain the identity. □